

# The String Deviation Equation

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February 7, 2008

Eprint: <http://xxx.lanl.gov/abs/gr-qc/9810043>

Comments: 18 pages 57580 bytes, no diagrams, LaTeX2e.

2 KEYPHRASES:

String Deviation: Geodesic Deviation.

1999 PACS Classification Scheme:

<http://publish.aps.org/eprint/gateway/pacslist>

12.25+e

1991 Mathematics Subject Classification:

<http://www.ams.org/msc>

81T30.

## Abstract

It is well known that the relative motion of many particles can be described by the geodesic deviation equation. Less well known is that the geodesic deviation equation can be derived from the second covariant variation of the point particle's action. Here it is shown that the second covariant variation of the string action leads to a string deviation equation. This equation is a candidate for describing the relative motion of many strings, and can be reduced to the geodesic deviation equation. Like the geodesic deviation equation, the string deviation equation is also expressible in terms of momenta and projecta. It is also shown that a combined action exists, the first variation of which gives the deviation equations. The combined actions allow the deviation equations to be expressed solely in terms of the Riemann tensor, the coordinates, and momenta. In particular geodesic deviation can be expressed as:

$$\dot{\Pi}^\mu = R^\mu_{\cdot\alpha\beta\gamma} r^\gamma P^\beta \dot{x}^\alpha,$$

and string deviation can be expressed as:

$$\dot{\Pi}_\tau^\mu + \Pi_\sigma'^\mu = R^\mu_{\cdot\alpha\beta\gamma} r^\gamma (P_\tau^\beta \dot{x}^\alpha + P_\sigma^\beta x'^\alpha).$$

## 1 Introduction

The second covariant variation of the point particle action produces the geodesic deviation equation Synge (1926) [1], Hawking and Ellis (1973) [2] and Bazanski (1977) [3]. Here the second variation method is applied to the string action Scherk (1974) [5] to produce a string deviation equation. Second order variations have been applied to the metric in quadratic gravitational theories Barth and Christensen (1983) [4]. Geodesic deviation equations are useful in investigating the structure of cosmological models Ellis and Van Elst (1997) [6], are used to study effective photon mass in higher order theory, Mohantz and Prasanna (1997) [7], are used to study the stability of Bianchi models, DiBari and Cipriani (1998) [8], and also are used to study the geometry of impulsive gravitational waves, Steinbauer (1998) [9] and Kunzinger and Steinbauer (1998) [10]. The equations for geodesic deviation sometimes can be solved by the inverse scattering method, Varlamov (1998) [11]. The string deviation equations might have application in studying objects such as binary cosmic strings, DasGupta and Rohm (1996) [12], cosmic strings with wiggles, Kim and Sikivie (1994) [13], string interaction and collision, Letelier et al (1993) [14], and string evolution, Ausin, Copeland, and Kibble (1993) [15]. Also they might have application to the loop space approach to fundamental strings, Bowick and Rajeev (1987) [16]. The string deviation equation has one separation vector which connects adjacent strings. Both deviation equations can be derived from combined actions as is shown here in section 4. At least five problems which are left for future investigation include the following. *Firstly* the geometrical interpretation of the string deviation equation is not looked at in detail. Reduction of the string deviation equation gives the geodesic deviation equation. Both deviation equations are expressible solely in terms of momenta and projecta. These can be defined for both arbitrary particle and string Lagrangians and specifically for the standard case. *Secondly* the production of higher order deviation equations corresponding to the third covariant variation, fourth covariant variation and so on ... *Thirdly* the construction of momenta and phase space with a view to quantizing the system by traditional canonical methods: this has been done for the geodesic deviation equation Roberts (1996) [17]. *Fourthly* the implication for the algebra obtained by Fourier transforming string modes. *Fifthly* the application to hypothetical physical strings such as cosmic strings and fundamental strings.

There are at least three ways of generalizing the point particles action. The **first** is by changing the number of dimensions of the ambient space or

the number of of the intrinsic space, changing the dimensions of the intrinsic space one has strings and membranes. The **second** is by grading the action with fermionic charges. The **third** is to produce deviation equations. Such generalizations can be done in combination and in principle the deviation method used here can be applied to the actions corresponding to other generalizations, again these points are left for future investigation. A point particle which carries charge Roberts (1989) [18] cannot be constructed from a Lagrangian theory because none of the Lorentz-Dirac, Hobbs, and DeWitt-Brehme terms cannot be recovered, resulting in no Lagrange method being useable here. Perhaps this is because only *part* of the physical system is being described, quantum electrodynamics describes the *full* system and this is Lagrangian based. The conventions used are the same as in Hawking and Ellis (1973)[2] (in particular the signature is  $-, +, +, +$ ), additionally  $\dot{x}^2 = \dot{x} \cdot \dot{x} = \dot{x}^\alpha \dot{x}_\alpha$ .

## 2 The First Covariant Variation.

### 2.1 The Point Particle

The action of a point particle in coordinate space is taken to be

$$S = \int_{\tau_1}^{\tau_2} d\tau \mathcal{L}. \quad (1)$$

The standard point particle Lagrangian  $\mathcal{L}$  is given by

$$\mathcal{L} = -m \times \ell^n, \quad (2)$$

where the “length”  $\ell$  is

$$\ell \equiv \sqrt{-\dot{x} \cdot \dot{x}} = \sqrt{-g_{\alpha\beta} \frac{Dx^\alpha}{d\tau} \frac{Dx^\beta}{d\tau}}, \quad (3)$$

and  $\tau$  is the evolution parameter,  $m$  is the mass, and the velocity is  $\dot{x}^\alpha = Dx^\alpha/d\tau$ . The action with  $n = 1$  is reparameterization invariant, other values of  $n$  represent gauged-fixed actions and are not fully equivalent (e.g., they have different interpretations in the quantum theory of relativistic particles). Here the  $n = 1$  reparameterization invariant action is used. Varying the velocity one can interchange the variation and the covariant derivative thus

$$\delta \dot{x}^\alpha = \delta \frac{Dx^\alpha}{d\tau} = \frac{D\delta x^\alpha}{d\tau}, \quad (4)$$

c.f. [3] equation(1.4). Now varying the action

$$\delta S = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \delta x^\mu \Big|_{\tau_1}^{\tau_2} - \int_{\tau_1}^{\tau_2} d\tau \delta x^\mu \frac{D}{d\tau} \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}. \quad (5)$$

The first term of the action can be made to vanish by choosing the boundary condition

$$\delta x^\mu \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = 0, \quad (6)$$

at  $\tau_1$  and  $\tau_2$ , or more simply

$$\delta x^\mu \Big|_{\tau_1} = \delta x^\mu \Big|_{\tau_2} = 0, \quad (7)$$

The second term of 5 vanishes when the equation of motion

$$\frac{D}{d\tau} \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = 0, \quad (8)$$

is obeyed. Specifically for the standard Lagrangian 2

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \frac{m \dot{x}^\mu}{\ell}, \quad (9)$$

so that the equation of motion becomes the geodesic equation

$$\frac{D}{d\tau} \frac{\dot{x}^\mu}{\ell} = 0. \quad (10)$$

The momentum is defined by

$$P_\mu \equiv \frac{\delta S}{\delta \dot{x}^\mu} = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}, \quad (11)$$

the equality holding because the Lagrangians considered here are both explicitly defined and are only functions of  $\dot{x}^\mu$  not  $\ddot{x}^\mu$ . The derivative of the momenta is defined by

$$\dot{P}^\mu \equiv \frac{D}{d\tau} P^\mu. \quad (12)$$

In terms of the momentum the boundary condition 6 is

$$\delta x^\mu P_\mu = 0, \quad (13)$$

and the equation of motion 8 is

$$\dot{P}^\mu = 0. \quad (14)$$

Here throughout momenta are introduced last so as to emphasise that the actions considered are coordinate space actions rather than phase space actions.

## 2.2 The String

The action of a string [5] eq.I.16 is

$$S = \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \mathcal{L}, \quad (15)$$

where the standard Lagrangian  $\mathcal{L}$  is given by

$$\mathcal{L} = -\frac{\mathcal{A}}{2\pi\alpha'} \quad (16)$$

and the “area”  $\mathcal{A}$  is

$$\begin{aligned} \mathcal{A} &\equiv \sqrt{(\dot{x} \cdot x')^2 - \dot{x}^2 x'^2} \\ &= \sqrt{g_{\alpha\beta} \frac{Dx^\alpha}{d\tau} \frac{Dx^\beta}{d\sigma} g_{\gamma\delta} \frac{Dx^\gamma}{d\tau} \frac{Dx^\delta}{d\sigma} - g_{\alpha\beta} \frac{Dx^\alpha}{d\tau} \frac{Dx^\beta}{d\tau} g_{\gamma\delta} \frac{Dx^\gamma}{d\sigma} \frac{Dx^\delta}{d\sigma}}, \end{aligned} \quad (17)$$

and  $\tau$  is the evolution parameter,  $\sigma$  is the kinematic parameter,  $\alpha'$  is the string tension,  $\dot{x}^\alpha \equiv \frac{Dx^\alpha}{d\tau}$ ,  $x'^\alpha \equiv \frac{Dx^\alpha}{d\sigma}$ , and the absolute derivatives are  $\frac{D}{d\tau} \equiv \dot{x}^\alpha \nabla_\alpha$ ,  $\frac{D}{d\sigma} \equiv x'^\alpha \nabla_\alpha$ . Varying the velocities in a similar manner to 4 gives

$$\begin{aligned} \delta \dot{x}^\alpha &= \frac{D}{d\tau} \delta x^\alpha, \\ \delta x'^\alpha &= \frac{D}{d\sigma} \delta x^\alpha. \end{aligned} \quad (18)$$

Now varying the action c.f. [5] eqs.I.17-19

$$\begin{aligned} \delta S = & - \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \delta x^\mu \left( \frac{D}{d\tau} \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} + \frac{D}{d\sigma} \frac{\partial \mathcal{L}}{\partial x'^\mu} \right) \\ & + \int_0^\pi d\sigma \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \delta x^\mu \Big|_{\tau=\tau_1}^{\tau=\tau_2} \\ & + \int_{\tau_1}^{\tau_2} d\tau \frac{\partial \mathcal{L}}{\partial x'^\mu} \delta x^\mu \Big|_{\sigma=0}^{\sigma=\pi}. \end{aligned} \quad (19)$$

Choosing initial and final positions on the string to be fixed so that 7 is obeyed the second term vanishes. The third term vanishes when the edge condition, c.f. Scherk (1975) [5] eq.I.18.1

$$\frac{\partial \mathcal{L}}{\partial x'^\mu} \Big|_{\sigma=0} = \frac{\partial \mathcal{L}}{\partial x'^\mu} \Big|_{\sigma=\pi}, \quad (20)$$

is obeyed. This edge condition is for open strings, for closed string  $x_\mu(\tau, \sigma + 2\pi) = x_\mu(\tau, \sigma)$ , c.f. Scherk (1975) [5] section II.7. The vanishing of the first term gives the equation of motion

$$\frac{D}{d\tau} \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} + \frac{D}{d\sigma} \frac{\partial \mathcal{L}}{\partial x'^\mu} = 0. \quad (21)$$

Using the reduction equation

$$\frac{\partial \mathcal{L}}{\partial x'^\alpha} = 0, \quad (22)$$

the string equation of motion 21 takes the same form as point particle equation of motion 8. The reduction equations

$$\dot{x} \cdot x' = 0, \quad x'^2 = +1, \quad (23)$$

reduce the “area” 17 to the “length” 3. The reduction equation

$$m = (2\pi\alpha')^{-1}, \quad (24)$$

equates the coupling constants so that the point particle equation of motion 8 is recovered. Again one can define momenta

$$\begin{aligned} P_{\tau\mu} &\equiv \frac{\delta S}{\delta \dot{x}^\mu} = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}, \\ P_{\sigma\mu} &\equiv \frac{\delta S}{\delta x'^\mu} = \frac{\partial \mathcal{L}}{\partial x'^\mu}. \end{aligned} \quad (25)$$

Derivatives of the momenta are defined by

$$\begin{aligned} \dot{P}_\tau^\mu &\equiv \frac{D}{d\tau} P_\tau^\mu, \\ P_\sigma'^\mu &\equiv \frac{D}{d\sigma} P_\sigma^\mu. \end{aligned} \quad (26)$$

These momenta allow the edge condition 20 to be put in the form

$$P_\sigma^\mu |_{\sigma=0} = P_\sigma^\mu |_{\sigma=\pi}, \quad (27)$$

and equation of motion takes the simple form

$$\dot{P}_\tau^\mu + P_\sigma'^\mu = 0. \quad (28)$$

For the specific Lagrangian 16 the momenta 25 are

$$\begin{aligned} P_\tau^\mu &= \frac{-1}{2\pi\alpha'\mathcal{A}} \left( (\dot{x} \cdot x') x'^\mu - x'^2 \dot{x}^\mu \right), \\ P_\sigma^\mu &= \frac{-1}{2\pi\alpha'\mathcal{A}} \left( (\dot{x} \cdot x') \dot{x}^\mu - \dot{x}^2 x'^\mu \right). \end{aligned} \quad (29)$$

The first reduction equation 22 becomes

$$\dot{x}^2 x'^\mu = 0 \quad (30)$$

### 3 The Second Covariant Variation.

#### 3.1 The Point Particle Again

The second variation of the point particle action is

$$\begin{aligned} \delta^2 S &= \int_{\tau_1}^{\tau_2} d\tau \delta(\delta\mathcal{L}) \\ &= \int_{\tau_1}^{\tau_2} d\tau \delta\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} \delta\dot{x}^\mu\right) \\ &= \int_{\tau_1}^{\tau_2} d\tau \left( \frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} \delta^2 \dot{x}^\mu - \frac{\partial^2\mathcal{L}}{\partial\dot{x}^\nu \partial\dot{x}^\mu} \delta\dot{x}^\mu \delta\dot{x}^\nu \right). \end{aligned} \quad (31)$$

The Ricci identity for  $\dot{x}^\alpha$  is

$$\delta^2 \dot{x}^\alpha = \frac{D}{D\tau} \delta^2 x^\alpha + R_{\mu\nu\rho}^\alpha \delta x^\mu \delta x^\nu \dot{x}^\rho, \quad (32)$$

c.f. [3] eq.(2.2). Inserting 32 into 31 gives the second covariant variation

$$\begin{aligned} \delta^2 S &= - \int_{\tau_1}^{\tau_2} d\tau \delta x^\nu \left[ \frac{D}{d\tau} \left( \frac{\partial^2\mathcal{L}}{\partial\dot{x}^\nu \partial\dot{x}^\mu} \frac{D}{d\tau} \delta x^\mu \right) - \frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} R_{\alpha\nu\rho}^\mu \delta x^\alpha \dot{x}^\rho \right] \\ &\quad - \int_{\tau_1}^{\tau_2} d\tau \delta^2 x^\mu \frac{D}{d\tau} \frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} \\ &\quad \left( \delta^2 x^\mu \frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} + \frac{\partial^2\mathcal{L}}{\partial\dot{x}^\nu \partial\dot{x}^\mu} \delta \frac{D}{d\tau} (\delta x^\mu) \delta x^\nu \right) \Big|_{\tau=\tau_1}^{\tau=\tau_2}. \end{aligned} \quad (33)$$

The last term vanishes by the first order boundary condition 7. The second to last term vanishes by the second order boundary condition

$$\delta^2 x^\mu \Big|_{\tau_2} = \delta^2 x^\mu \Big|_{\tau_1} = 0, \quad (34)$$

alternatively it vanishes by the equation of motion 8. There remains the first integral term. **If** the infinitesimal  $\delta x^\alpha$  inside the square bracket is **identified** with the separation vector  $r^\alpha$ ; then 33 vanishes when the deviation equation

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} R^\mu_{\alpha\nu\beta} r^\alpha \dot{x}^\beta - \frac{D}{d\tau} \left( \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^\nu \partial \dot{x}^\mu} \frac{D}{d\tau} r^\mu \right) = 0, \quad (35)$$

is obeyed. Introducing the momentum 11 and the general projection

$$H_{\mu\nu} \equiv \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^\nu \partial \dot{x}^\mu} \quad (36)$$

the point particle deviation equation is

$$P^\mu r^\alpha \dot{x}^\beta R_{\mu\alpha\nu\beta} - \frac{D}{d\tau} (H^\nu_\mu \dot{r}^\mu) = 0, \quad (37)$$

where  $\dot{r}^\mu$  is defined by

$$\dot{r}^\mu \equiv \frac{D}{d\tau} r^\mu. \quad (38)$$

The specific Lagrangian 2 has first order variation 9 and the second order variation gives the general projection 36 explicitly as

$$H^{\mu\nu} = \frac{m}{\ell} h^{\mu\nu}, \quad (39)$$

where the standard projection tensor  $h^{\mu\nu}$  is defined by

$$h^{\mu\nu} \equiv g^{\mu\nu} + \frac{1}{\ell^2} \dot{x}^\mu \dot{x}^\nu, \quad (40)$$

and has trace  $h^\mu_\mu = d - 1$ , and  $d$  is the dimension of the spacetime. Substituting for  $P^\mu$  using equation 11 and  $H^{\mu\nu}$  using equation 39, multiplying by  $m\ell$  and then using the algebraic properties of the Riemann tensor gives the geodesic deviation equation

$$R^\mu_{\alpha\beta\gamma} \dot{x}^\alpha \dot{x}^\gamma r^\beta + \ell \frac{D}{d\tau} \left( \frac{1}{\ell} h^\mu_\alpha \dot{r}^\alpha \right) = 0. \quad (41)$$

### 3.2 The String Again

For the string action the second variation gives six terms

$$\begin{aligned}
\delta^2 S &= \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \delta \left[ \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \partial \dot{x}^\mu + \frac{\partial \mathcal{L}}{\partial x'^\mu} \delta x'^\mu \right] \\
&= \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \left[ \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \delta^2 \dot{x}^\mu + \frac{\partial \mathcal{L}}{\partial x'^\mu} \delta^2 x'^\mu \right. \\
&\quad + \delta \left( \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \right) \delta \dot{x}^\mu + \delta \left( \frac{\partial \mathcal{L}}{\partial x'^\mu} \right) \delta x'^\mu \left. \right] \\
&= \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \left[ \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \delta^2 \dot{x}^\mu + \frac{\partial \mathcal{L}}{\partial x'^\mu} \delta^2 x'^\mu \right. \\
&\quad + \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^\nu \partial \dot{x}^\mu} \delta \dot{x}^\mu \delta \dot{x}^\nu + \frac{\partial^2 \mathcal{L}}{\partial x'^\nu \partial \dot{x}^\mu} \delta \dot{x}^\mu \delta x'^\nu \\
&\quad + \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^\nu \partial x'^\mu} \delta x'^\mu \delta \dot{x}^\nu + \left. \frac{\partial^2 \mathcal{L}}{\partial x'^\nu \partial x'^\mu} \delta x'^\mu \delta x'^\nu \right]. \tag{42}
\end{aligned}$$

In addition to the Ricci identity 32 for  $\dot{x}^\alpha$  use the Ricci identity for  $x'^\alpha$

$$\delta^2 x'^\alpha = \frac{D}{d\sigma} \delta^2 x^\alpha + R_{\mu\nu\rho}^\alpha \delta x^\mu x'^\nu x'^\rho, \tag{43}$$

to give the twice varied action

$$\begin{aligned}
\delta^2 S &= \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \left[ \frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} \frac{D}{d\tau} \delta^2 x^\alpha + \frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} R_{\mu\nu\rho}^\alpha \delta x^\mu x'^\nu \dot{x}^\rho \right. \\
&\quad + \frac{\partial \mathcal{L}}{\partial x'^\alpha} \frac{D}{d\sigma} \delta^2 x^\alpha + \frac{\partial \mathcal{L}}{\partial x'^\alpha} R_{\mu\nu\rho}^\alpha \delta x^\mu \delta x'^\nu x'^\rho \\
&\quad + \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^\nu \partial \dot{x}^\mu} \delta \dot{x}^\mu \frac{D}{d\tau} \delta x^\nu + \frac{\partial^2 \mathcal{L}}{\partial x'^\nu \partial \dot{x}^\mu} \delta \dot{x}^\mu \frac{D}{d\sigma} \delta x^\nu \\
&\quad + \left. \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^\nu \partial x'^\mu} \delta x'^\mu \frac{D}{d\tau} \delta x^\nu + \frac{\partial^2 \mathcal{L}}{\partial x'^\nu \partial x'^\mu} \delta x'^\mu \frac{D}{d\sigma} \delta x^\nu \right]. \tag{44}
\end{aligned}$$

The first and third terms vanish if the first order equations of motion are assumed and also the second order boundary conditions

$$\delta^2 x^\alpha \big|_{\tau_1} = \delta^2 x^\alpha \big|_{\tau_2} = \delta^2 x^\alpha \big|_{\sigma=0} = \delta^2 x^\alpha \big|_{\sigma=\pi} = 0. \tag{45}$$

leaving

$$\begin{aligned}
\delta^2 S = & \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \left( \frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} \delta \dot{x}^\alpha + \frac{\partial \mathcal{L}}{\partial x'^\alpha} \delta x'^\alpha \right) \delta x^\mu \delta x^\nu R_{\cdot\mu\nu\rho}^\alpha \\
& + \left[ \int_0^\pi d\sigma \left( \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^\nu \partial \dot{x}^\mu} \delta \dot{x}^\mu + \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^\nu \partial x'^\mu} \delta x'^\mu \right) \delta x^\nu \right] \Big|_{\tau_1}^{\tau_2} \\
& - \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \delta x^\nu \frac{D}{d\tau} \left( \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^\nu \partial \dot{x}^\mu} \delta \dot{x}^\mu + \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^\nu \partial x'^\mu} \delta x'^\mu \right) \\
& + \left[ \int_{\tau_1}^{\tau_2} d\tau \left( \frac{\partial^2 \mathcal{L}}{\partial x'^\nu \partial \dot{x}^\mu} \delta \dot{x}^\mu + \frac{\partial^2 \mathcal{L}}{\partial x'^\nu \partial x'^\mu} \delta x'^\mu \right) \delta x'^\nu \right] \Big|_{\sigma=0}^{\sigma=\pi} \\
& - \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \delta x^\nu \frac{D}{d\sigma} \left( \frac{\partial^2 \mathcal{L}}{\partial x'^\nu \partial \dot{x}^\mu} \delta \dot{x}^\mu + \frac{\partial^2 \mathcal{L}}{\partial x'^\nu \partial x'^\mu} \delta x'^\mu \right). \tag{46}
\end{aligned}$$

The second to last term vanishes by the edge condition 20 of the first variation. The second term vanishes by the condition 7 leaving

$$\begin{aligned}
\delta^2 S = & \int_{\tau_1}^{\tau_2} \int_0^\pi d\sigma \delta x^\nu \left[ \left( \frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} \delta \dot{x}^\beta + \frac{\partial \mathcal{L}}{\partial x'^\alpha} \delta x'^\beta \right) \delta x^\mu R_{\cdot\mu\nu\rho}^\alpha \right. \\
& - \frac{D}{d\tau} \left( \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^\nu \partial \dot{x}^\mu} \delta \dot{x}^\mu + \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^\nu \partial x'^\mu} \delta x'^\mu \right) \\
& \left. - \frac{D}{d\sigma} \left( \frac{\partial^2 \mathcal{L}}{\partial x'^\nu \partial \dot{x}^\mu} \delta \dot{x}^\mu + \frac{\partial^2 \mathcal{L}}{\partial x'^\nu \partial x'^\mu} \delta x'^\mu \right) \right]. \tag{47}
\end{aligned}$$

Now  $\delta x^\mu$  is **identified** with the separation vector  $r^\mu$ . No rigorous proof that the indentification holds by necessity is given here. There are three justifications for this identification. *Firstly*, this identification is ANALOGOUS to that of the geodesic case. *Secondly*, there is no ALTERNATIVE vector which could be used except ones differentiated with respect to either  $\tau$  or  $\sigma$ . *Thirdly*, POST-HOC it produces simple equations.

$$\begin{aligned}
& \left( \frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} \dot{x}^\beta + \frac{\partial \mathcal{L}}{\partial x'^\alpha} \delta x'^\beta \right) r^\mu R_{\cdot\mu\nu\beta}^\alpha \\
& - \frac{D}{d\tau} \left( \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^\nu \partial \dot{x}^\mu} \frac{D}{d\tau} r^\mu + \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^\nu \partial x'^\mu} \frac{D}{d\sigma} r^\mu \right) \\
& - \frac{D}{d\sigma} \left( \frac{\partial^2 \mathcal{L}}{\partial x'^\nu \partial \dot{x}^\mu} \frac{D}{d\tau} r^\mu + \frac{\partial^2 \mathcal{L}}{\partial x'^\nu \partial x'^\mu} \frac{D}{d\sigma} r^\mu \right) = 0. \tag{48}
\end{aligned}$$

Using the first reduction equation 22 and also the second order reduction equations

$$\frac{\partial^2 \mathcal{L}}{\partial \dot{x}^\nu \partial x'^\mu} = \frac{\partial^2 \mathcal{L}}{\partial x'^\nu \partial \dot{x}^\mu} = 0, \quad (49)$$

and

$$\frac{\partial^2 \mathcal{L}}{\partial x'^\nu \partial x'^\mu} = 0, \quad (50)$$

the deviation equation 35 is recovered. Define the general projecta

$$\begin{aligned} H_{\tau\tau\mu\nu} &\equiv \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^\mu \partial \dot{x}^\nu}, \\ H_{\tau\sigma\mu\nu} &\equiv \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^\mu \partial x'^\nu}, \\ H_{\sigma\tau\mu\nu} &\equiv \frac{\partial^2 \mathcal{L}}{\partial x'^\mu \partial x'^\nu}, \\ H_{\sigma\sigma\mu\nu} &\equiv \frac{\partial^2 \mathcal{L}}{\partial x'^\mu \partial x'^\nu}, \end{aligned} \quad (51)$$

note that  $H_{\tau\sigma}^{\mu\nu}$  does not equal  $H_{\sigma\tau}^{\mu\nu}$  unless the matrix is symmetric. The four projecta are best expressed in terms of their symmetric and antisymmetric parts, thus:  $H_{\tau\tau}^{\mu\nu}$ ,  $H_{\sigma\sigma}^{\mu\nu}$ ,  $H_{\tau\sigma}^{(\mu\nu)}$ , and  $H_{\tau\sigma}^{[\mu\nu]}$ . Using the momenta 25, the string deviation equation is

$$\begin{aligned} &\left( P_\tau^\alpha \dot{x}^\beta + P_\sigma^\alpha x'^\beta \right) r^\mu R_{\alpha\mu\nu\beta} \\ &- \frac{D}{d\tau} \left( H_{\tau\tau}^{\nu\mu} \dot{r}_\mu + H_{\tau\sigma}^{\nu\mu} r'_\mu \right) - \frac{D}{d\sigma} \left( H_{\sigma\tau}^{\nu\mu} \dot{r}_\mu + H_{\sigma\sigma}^{\nu\mu} r'_\mu \right) = 0. \end{aligned} \quad (52)$$

For the standard Lagrangian 16 the projection tensors become

$$\begin{aligned} H_{\tau\tau}^{\mu\nu} &= \frac{x'^2}{2\pi\alpha' \mathcal{A}} h^{\mu\nu}, \\ H_{\sigma\sigma}^{\mu\nu} &= \frac{\dot{x}^2}{2\pi\alpha' \mathcal{A}} h^{\mu\nu}, \\ H_{\tau\sigma}^{(\mu\nu)} &= \frac{-(\dot{x} \cdot x') h^{\mu\nu}}{2\pi\alpha' \mathcal{A}} \\ H_{\tau\sigma}^{[\mu\nu]} &= \frac{1}{2\pi\alpha' \mathcal{A}} (-\dot{x}^\mu x'^\nu + x'^\mu \dot{x}^\nu), \end{aligned} \quad (53)$$

where the standard projection tensor is defined by

$$h^{\mu\nu} \equiv g^{\mu\nu} + \frac{1}{\mathcal{A}^2}(\dot{x}^2 x'^\mu x'^\nu - (\dot{x} \cdot x')(\dot{x}^\mu x'^\nu + x'^\mu \dot{x}^\nu) + x'^2 \dot{x}^\mu \dot{x}^\nu), \quad (54)$$

and has trace  $h^\mu_\mu = d-2$ . Using the reduction equations 23,30 this projection tensor reduces to 40. In the specific case of the standard Lagrangian 16, multiplying by  $-2\pi\alpha'\mathcal{A}$  and substituting gives the standard string deviation

$$\begin{aligned} & \mathcal{A}(g^{\alpha\beta} - h^{\alpha\beta})r^\mu R_{\alpha\mu\nu\beta} \\ & - \frac{D}{d\tau} \frac{1}{\mathcal{A}} \left( h^\nu_\mu (x'^2 \dot{r}^\mu - \dot{x} \cdot x' r'^\mu) + r'^\mu (\dot{x}^\mu x'^\nu - x'^\mu \dot{x}^\nu) \right) \\ & - \frac{D}{d\sigma} \frac{1}{\mathcal{A}} \left( h^\nu_\mu (-\dot{x} \cdot x' \dot{r}^\mu + \dot{x}^2 r'^\mu) + \dot{r}^\mu (-\dot{x}^\mu x'^\nu + x'^\mu \dot{x}^\nu) \right) = 0. \end{aligned} \quad (55)$$

For this case the reduction equation 49 is satisfied identically and 50 becomes

$$\frac{D}{d\sigma} \frac{1}{\mathcal{A}} h^\nu_\mu \dot{x}^2 r'^\mu = 0. \quad (56)$$

Applying the reduction equations 23,30, and 56 to the standard string deviation 55 gives the geodesic deviation equation 41.

## 4 The Combined Action.

### 4.1 The Point Particle Yet Again

The equations produced from a second covariant variation can often be derived from a combined action. For the point particle consider

$$S_c = \int_{\tau_1}^{\tau_2} d\tau \frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} \frac{D}{d\tau} r^\alpha, \quad (57)$$

with  $\tau$  the evolution parameter and  $r^\alpha$  a separation vector. The Ricci identity can be expressed as

$$\delta \left( \frac{D}{d\tau} r^\alpha \right) = \frac{D}{d\tau} \delta r^\alpha + R^\alpha_{\lambda\mu\nu} r^\lambda \delta x^\mu \dot{x}^\nu, \quad (58)$$

Varying the action gives

$$\begin{aligned} \delta S_c &= \int_{\tau_1}^{\tau_2} d\tau \left[ \delta \left( \frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} \right) \frac{D}{d\tau} r^\alpha + \frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} R^\alpha_{\lambda\mu\nu} r^\lambda \delta x^\mu \dot{x}^\nu \right] \\ &+ \left[ \frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} \delta r^\alpha \right]_{\tau_1}^{\tau_2} - \int_{\tau_1}^{\tau_2} d\tau \delta r^\alpha \frac{D}{d\tau} \frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha}, \end{aligned} \quad (59)$$

Assuming that  $\mathcal{L}$  is only an explicit function of  $\dot{x}^\alpha$  and not of  $x^\alpha, r^\alpha, \dot{r}^\alpha$  gives

$$\begin{aligned}\delta S_c &= \int_{\tau_1}^{\tau_2} d\tau \left[ -\frac{D}{d\tau} \left( \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^\beta \partial \dot{x}^\alpha} \frac{D}{d\tau} r^\alpha \right) + \frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} R_{\lambda\beta\nu}^\alpha r^\lambda \dot{x}^\nu \right] \delta x^\beta \\ &\quad - \int_{\tau_1}^{\tau_2} d\tau \delta r \frac{D}{d\tau} \frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} \\ &\quad + \left[ \delta x^\beta \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^\beta \partial \dot{x}^\alpha} \frac{D}{d\tau} r^\alpha + \frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} \delta r^\alpha \right] \Big|_{\tau_1}^{\tau_2} .\end{aligned}\tag{60}$$

The boundary term vanishes if

$$\delta x^\alpha \Big|_{\tau_1} = \delta x^\alpha \Big|_{\tau_2} = \delta r^\alpha \Big|_{\tau_1} = \delta r^\alpha \Big|_{\tau_2} = 0.\tag{61}$$

The  $\delta r^\alpha$  integral term vanishes if the particle equation of motion 8 is obeyed, the  $\delta x^\alpha$  integral term vanishes if the particle deviation equation 35 is obeyed. Defining momenta

$$\begin{aligned}P^\mu &\equiv \frac{\delta S_c}{\delta \dot{r}^\mu} = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}, \\ \Pi^\mu &\equiv \frac{\delta S_c}{\delta \dot{x}^\mu} = \dot{r}^\nu \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^\mu \partial \dot{x}^\nu} = \dot{r}^\nu H_\nu^\mu,\end{aligned}\tag{62}$$

and

$$\dot{\Pi}^\mu \equiv \frac{D}{d\tau} \Pi^\mu,\tag{63}$$

the geodesic deviation equation 37 takes the simple form

$$\dot{\Pi}^\mu = R_{\alpha\beta\gamma}^\mu r^\gamma P^\beta \dot{x}^\alpha.\tag{64}$$

## 4.2 The String Yet Again

For the string consider the combined action

$$S_c = \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \left( \frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} \frac{D}{d\tau} r^\alpha + \frac{\partial \mathcal{L}}{\partial x'^\alpha} \frac{D}{d\sigma} r^\alpha \right)\tag{65}$$

Varying

$$\begin{aligned}\delta S_c &= \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \\ &\quad \{ -\delta r^\alpha \left( \frac{D}{d\tau} \frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} + \frac{D}{d\sigma} \frac{\partial \mathcal{L}}{\partial x'^\alpha} \right) + \delta x^\beta \left( \dot{x}^\nu \frac{\partial \mathcal{L}}{\partial \dot{x}^\nu} + x'^\nu \frac{\partial \mathcal{L}}{\partial x'^\nu} \right) R_{\lambda\beta\nu}^\alpha r^\lambda \}\end{aligned}$$

$$\begin{aligned}
& -\frac{D}{d\tau} \left( \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^\beta \partial \dot{x}^\alpha} \frac{D}{d\tau} r^\alpha + \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^\beta \partial x'^\alpha} \frac{D}{d\sigma} r^\alpha \right) \\
& -\frac{D}{d\sigma} \left( \frac{\partial^2 \mathcal{L}}{\partial x'^\beta \partial \dot{x}^\alpha} \frac{D}{d\tau} r^\alpha + \frac{\partial^2 \mathcal{L}}{\partial x'^\beta \partial x'^\alpha} \frac{D}{d\tau} r^\alpha \right) \\
& + \left[ \int_0^\pi d\sigma \left( \delta r^\alpha \frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} + \delta x^\beta \left( \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^\beta \partial \dot{x}^\alpha} \frac{D}{d\tau} r^\alpha + \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^\beta \partial x'^\alpha} \frac{D}{d\sigma} r^\alpha \right) \right) \right] \Big|_{\tau=\tau_1}^{\tau=\tau_2} \\
& + \left[ \int_{\tau_1}^{\tau_2} d\tau \left( \delta r^\alpha \frac{\partial \mathcal{L}}{\partial x'^\alpha} + \delta x^\beta \left( \frac{\partial^2 \mathcal{L}}{\partial x'^\beta \partial \dot{x}^\alpha} \frac{D}{d\tau} r^\alpha + \frac{\partial^2 \mathcal{L}}{\partial x'^\beta \partial x'^\alpha} \frac{D}{d\sigma} r^\alpha \right) \right) \right] \Big|_{\sigma=0}^{\sigma=\pi}. \quad (66)
\end{aligned}$$

The first integral term vanishes by the string equation of motion 21. The second integral term vanishes by the string deviation equation 48. The boundary terms vanish if 61 is obeyed. Using the general projecta 51 and defining

$$\begin{aligned}
P_{\tau\mu} &\equiv \frac{\delta S_c}{\delta \dot{r}^\mu} = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}, \\
P_{\sigma\mu} &\equiv \frac{\delta S_c}{\delta r'^\mu} = \frac{\partial \mathcal{L}}{\partial x'^\mu}, \\
\Pi_{\tau\mu} &\equiv \frac{\delta S_c}{\delta \dot{x}^\mu} = \dot{r}_\alpha H_{\tau\tau}^{\mu\nu} + r'_\alpha H_{\tau\sigma}^{\mu\nu}, \\
\Pi_{\sigma\mu} &\equiv \frac{\delta S_c}{\delta x'^\mu} = \dot{r}_\alpha H_{\sigma\tau}^{\mu\alpha} + r'_\alpha H_{\sigma\sigma}^{\mu\alpha}, \quad (67)
\end{aligned}$$

and substituting into 52 the string deviation equation takes the simple form

$$\dot{\Pi}_\tau^\mu + \Pi_\sigma'^\mu = R_{\alpha\beta\gamma}^\mu r^\gamma (P_\tau^\beta \dot{x}^\alpha + P_\sigma^\beta x'^\alpha). \quad (68)$$

## 5 Conclusion.

A point particle obeys the geodesic equation 8 and many particles have relative motion described by the geodesic deviation equation 35. A single string obeys the equation of motion 21 and second order variation of the action gives the string deviation equation 48. For the standard point particle Lagrangian 2 the geodesic equation is 10 and the geodesic deviation equation is 41; for the standard string Lagrangian 16 the equation of motion is 21 and the deviation equation is 55. The second order variations can be expressed

in combined first order form, this leads to the deviation equations of the abstract. The string deviation equation is a candidate for the description of the relative motion of many strings. The combined actions of section 4 describe a new gauge theory to which the full paraphernalia of gauge theory can be applied.

## 6 Acknowledgements

I would like to thank T.W.B.Kibble for his interest in this work, and also A.L.Larsen for pointing out similar work on perturbing strings by himself and Frolov [21], Carter [19], and Guven [20]. This work has been supported in part by the South African National Research Foundation, formally called the Foundation for Research and Development (FRD).

## References

- [1] Synge,J.L.(1926) ♣ 1.1  
On the first and second variations of the line integral in Riemannian geometry.  
*Proc.London Math.Soc.*,**25**,247.
- [2] Hawking, S. W. and Ellis, G. F. R. (1973) ♣ 1.1  
*The large scale structure of space-time*.  
Math.Rev.**54**#12154  
Cambridge University Press. page 108.
- [3] Bazanski, S. L. (1977) ♣ 1.1  
Kinematic of Relative Motion of Test Particles in General Relativity.  
Math.Rev.**75**#8867  
*Ann. Inst. Henri Poincare*, **XXVII**, 115.
- [4] Barth,N.H. and Christensen,S.M.(1983) ♣ 1.1  
Quantizing fourth-order gravity theories: the functional integral.  
Math.Rev.84j:83032  
*Phys.Rev.D*,**28**,1876,  
especially Appendices B and D.
- [5] Scherk, J. (1974) ♣ 1.1  
An Introduction to the Theory of Dual Models and Strings.

Math.Rev.**56**#17602  
*Rev. Mod. Phys.*, **47**, 123.

- [6] Ellis, D. F. R. and Van Elst, H. (1997) ♣ 1.1  
 Deviation of Geodesics in FLRW Spacetime Geometries.  
 gr-qc/9709060  
 Contribution to Engelbert Schücking Festschrift
- [7] Mohantz,S. and Prasanna,A.R. (1997) ♣ 1.1  
 Geodesic Deviation of Photons in Einstein and Higher Derivative Gravity.  
 gr-qc/97010009 v3 17 Jan 1997.
- [8] DiBari,M. and Cipriani,P. (1998) ♣ 1.1  
 The Instability of Bianchi IX Dynamics: The Geodesic Deviation Equation in the Finsler Spaces.  
 gr-qc/9807022 9 July 1998.
- [9] Steinbauer,R.(1998) ♣ 1.1  
 Geodesics and Geodesic Deviation for Impulsive Gravitational Waves.  
 gr-qc/9710119  
*J.Math.Phys.*,**39**,2201.
- [10] Kunzinger,M. and Steinbauer,R. (1998) ♣ 1, 1  
 A Rigorous Solution Concept for Geodesic and Geodesic Deviation Equations in Impulsive Gravitational Waves.  
 gr-qc/9806009 3 June 1998.  
*J.Math.Phys.*,**40**(1999)1479-1489.
- [11] Varlamov,V.V.(1998) ♣ 1.1  
 Equation of geodesic deviation solvable by the inverse scattering transform.  
 solv-int/9808007 17 Aug.1998
- [12] DasGupta,I. and Rohm,R. (1997) ♣ 1.1  
 Binary String Dynamics.  
*Phys.Rev.D*, **55**, 2504.
- [13] Kim, J. and Sikivie, P. (1994) ♣ 1.1  
 Streching Wiggly Strings.  
 hep-ph/9405227, Math.Rev.98a:83115  
*Phys. Rev.D*, **50**, 7410.

- [14] Letelier, P.S., Gal'tsov, D.V., and Grats, Yu.V. (1993) ♣ 1.1  
 Post-linear formalism for gravitating strings: crossed straight string collisions.  
 Math.Rev.94a:83019  
*Ann.Phys.*, **224**, 90.
- [15] Ausitn, D., Copeland, E.J., and Kibble, T.W.B. (1993) ♣ 1.1  
 Evolution of Cosmic String Configurations.  
*Phys.Rev.D* **48**, 5594.
- [16] Bowick, M. J. and Rajeev, S. J. (1987) ♣ 1.1  
 String Theory as the Kähler Geometry of Loop Space.  
 Math.Rev.89a:22034  
*Phys. Rev. Lett*, **58**, 535.
- [17] Roberts, M. D. (1996) ♣ 1.1  
 The Quantization of Geodesic Deviation.  
 gr-qc/9903097, Math.Rev.97h:83016  
*Gen. Rel. Grav.*, **28**, 1385.
- [18] Roberts, M D. (1989) ♣ 1.2  
 The Motion of a Charged Particle in a Spacetime with a Conformal Metric.  
 Math.Rev.90b:83006  
*Class. Q. Grav.*, **6**, 419.
- [19] Carter, B. (1993) ♣ 6  
 Perturbation dynamics for membranes and strings goverened by the Dirac-Goto-Nambu action in curved space.  
 Math.Rev.95a:83090  
*Phys.Rev.D* **48**, 4835.
- [20] Guven, J. (1993) ♣ 6  
 Perturbations of a topological defect as a heory of coupled scalar fields in curved space interacting with an external vector potential.  
 gr-qc/9304033, Math.Rev.94k:83064  
*Phys.Rev.D* **48**, 5562.
- [21] Frolov, V.P. and Larsen, A.L. (1994) ♣ 6  
 Propagation of perturbations along strings.  
 hep-th/9303001, Math.Rev.95a:83090  
*Nucl.Phys.B* **414**, 129.